

ASYMPTOTIC PROPERTY FOR NONLINEAR PERTURBED FUNCTIONAL DIFFERENTIAL SYSTEMS

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ABSTRACT. This paper shows that the solutions to nonlinear perturbed functional differential system

$$y' = f(t, y) + \int_{t_0}^t g(s, y(s), Ty(s))ds + h(t, y(t))$$

have the asymptotic property by imposing conditions on the perturbed part $\int_{t_0}^t g(s, y(s), Ty(s))ds, h(t, y(t))$, and on the fundamental matrix of the unperturbed system $y' = f(t, y)$.

1. Introduction

Elaydi and Farran[8] introduced the notion of exponential asymptotic stability(EAS) which is a stronger notion than that of ULS. They investigated some analytic criteria for an autonomous differential system and its perturbed systems to be EAS. Brauer [2] studied the asymptotic behavior of solutions of nonlinear systems and perturbations of nonlinear systems by means of analogue of the variation of constants formula for nonlinear systems due to V.M. Alekseev[1]. Pachpatte[14] investigated the stability and asymptotic behavior of solutions of the functional differential equation. Gonzalez and Pinto[9] proved theorems which relate the asymptotic behavior and boundedness of the solutions of nonlinear differential systems. Choi et al.[6,7] examined Lipschitz and exponential asymptotic stability for nonlinear functional systems. Also, Goo[10,11], Goo et al. [12], and Choi and Goo[4,5] investigated Lipschitz and asymptotic stability for perturbed differential systems.

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In this paper we study the asymptotic property for solutions of the nonlinear differential systems. The method incorporating integral inequalities takes an important place among the methods developed for the qualitative analysis of solutions to linear and nonlinear system of differential equations.

2. preliminaries

We consider the nonlinear nonautonomous differential system

$$(2.1) \quad x' = f(t, x), \quad x(t_0) = x_0,$$

where $f \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $\mathbb{R}^+ = [0, \infty)$ and \mathbb{R}^n is the Euclidean n -space. We assume that the Jacobian matrix $f_x = \partial f / \partial x$ exists and is continuous on $\mathbb{R}^+ \times \mathbb{R}^n$ and $f(t, 0) = 0$. Also, we consider the perturbed differential system of (2.1)

$$(2.2) \quad y' = f(t, y) + \int_{t_0}^t g(s, y(s), Ty(s)) ds + h(t, y(t)), \quad y(t_0) = y_0,$$

where $g \in C(\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$, $g(t, 0, 0) = 0$, $h(t, 0) = 0$, and $T : C(\mathbb{R}^+, \mathbb{R}^n) \rightarrow C(\mathbb{R}^+, \mathbb{R}^n)$ is a continuous operator.

For $x \in \mathbb{R}^n$, let $|x| = (\sum_{j=1}^n x_j^2)^{1/2}$. For an $n \times n$ matrix A , define the norm $|A|$ of A by $|A| = \sup_{|x| \leq 1} |Ax|$.

Let $x(t, t_0, x_0)$ denote the unique solution of (2.1) with $x(t_0, t_0, x_0) = x_0$, existing on $[t_0, \infty)$. Then we can consider the associated variational systems around the zero solution of (2.1) and around $x(t)$, respectively,

$$(2.3) \quad v'(t) = f_x(t, 0)v(t), \quad v(t_0) = v_0$$

and

$$(2.4) \quad z'(t) = f_x(t, x(t, t_0, x_0))z(t), \quad z(t_0) = z_0.$$

The fundamental matrix $\Phi(t, t_0, x_0)$ of (2.4) is given by

$$\Phi(t, t_0, x_0) = \frac{\partial}{\partial x_0} x(t, t_0, x_0),$$

and $\Phi(t, t_0, 0)$ is the fundamental matrix of (2.3).

Before giving further details, we give some of the main definitions that we need in the sequel[8].

DEFINITION 2.1. The system (2.1) (the zero solution $x = 0$ of (2.1)) is called

(S) *stable* if for any $\epsilon > 0$ and $t_0 \geq 0$, there exists $\delta = \delta(t_0, \epsilon) > 0$ such that if $|x_0| < \delta$, then $|x(t)| < \epsilon$ for all $t \geq t_0 \geq 0$,

(AS) *asymptotically stable* if it is stable and if there exists $\delta = \delta(t_0) > 0$ such that if $|x_0| < \delta$, then $|x(t)| \rightarrow 0$ as $t \rightarrow \infty$,

(ULS) *uniformly Lipschitz stable* if there exist $M > 0$ and $\delta > 0$ such that $|x(t)| \leq M|x_0|$ whenever $|x_0| \leq \delta$ and $t \geq t_0 \geq 0$,

(EAS) *exponentially asymptotically stable* if there exist constants $K > 0$, $c > 0$, and $\delta > 0$ such that

$$|x(t)| \leq K |x_0| e^{-c(t-t_0)}, \quad 0 \leq t_0 \leq t$$

provided that $|x_0| < \delta$,

(EASV) *exponentially asymptotically stable in variation* if there exist constants $K > 0$ and $c > 0$ such that

$$|\Phi(t, t_0, x_0)| \leq K e^{-c(t-t_0)}, \quad 0 \leq t_0 \leq t$$

provided that $|x_0| < \infty$.

REMARK 2.2. [9] The last definition implies that for $|x_0| \leq \delta$

$$|x(t)| \leq K |x_0| e^{-c(t-t_0)}, \quad 0 \leq t_0 \leq t.$$

We give some related properties that we need in the sequel.

We need Alekseev formula to compare between the solutions of (2.1) and the solutions of perturbed nonlinear system

$$(2.5) \quad y' = f(t, y) + g(t, y), \quad y(t_0) = y_0,$$

where $g \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$ and $g(t, 0) = 0$. Let $y(t) = y(t, t_0, y_0)$ denote the solution of (2.5) passing through the point (t_0, y_0) in $\mathbb{R}^+ \times \mathbb{R}^n$.

The following is a generalization to nonlinear system of the variation of constants formula due to Alekseev [1].

LEMMA 2.3. [2] *Let x and y be a solution of (2.1) and (2.5), respectively. If $y_0 \in \mathbb{R}^n$, then for all $t \geq t_0$ such that $x(t, t_0, y_0) \in \mathbb{R}^n$, $y(t, t_0, y_0) \in \mathbb{R}^n$,*

$$y(t, t_0, y_0) = x(t, t_0, y_0) + \int_{t_0}^t \Phi(t, s, y(s)) g(s, y(s)) ds.$$

LEMMA 2.4. [3] *Let $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$ and $w(u)$ be nondecreasing in u , $u \leq w(u)$. Suppose that for some $c > 0$,*

$$\begin{aligned} u(t) \leq c + \int_{t_0}^t \lambda_1(s) u(s) ds + \int_{t_0}^t \lambda_2(s) w(u(s)) ds + \int_{t_0}^t \lambda_3(s) \int_{t_0}^s \lambda_4(\tau) w(u(\tau)) d\tau ds \\ + \int_{t_0}^t \lambda_5(s) \int_{t_0}^s \lambda_6(\tau) w(u(\tau)) d\tau ds, \quad 0 \leq t_0 \leq t. \end{aligned}$$

Then

$$u(t) \leq W^{-1} \left[W(c) + \int_{t_0}^t \left(\lambda_1(s) + \lambda_2(s) + \lambda_3(s) \int_{t_0}^s \lambda_4(\tau) d\tau \right. \right. \\ \left. \left. + \lambda_5(s) \int_{t_0}^s \lambda_6(\tau) d\tau \right) ds \right],$$

where $t_0 \leq t < b_1$, $W(u) = \int_{u_0}^u \frac{ds}{w(s)}$, $W^{-1}(u)$ is the inverse of $W(u)$, and

$$b_1 = \sup \left\{ t \geq t_0 : W(c) + \int_{t_0}^t \left(\lambda_1(s) + \lambda_2(s) + \lambda_3(s) \int_{t_0}^s \lambda_4(\tau) d\tau \right. \right. \\ \left. \left. + \lambda_5(s) \int_{t_0}^s \lambda_6(\tau) d\tau \right) ds \in \text{dom} W^{-1} \right\}.$$

For the proof we need the following corollary from Lemma 2.4.

COROLLARY 2.5. *Let $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4 \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$ and $w(u)$ be nondecreasing in u , $u \leq w(u)$. Suppose that for some $c > 0$ and $0 \leq t_0 \leq t$,*

$$u(t) \leq c + \int_{t_0}^t \lambda_1(s) u(s) ds + \int_{t_0}^t \lambda_2(s) w(u(s)) ds \\ + \int_{t_0}^t \lambda_3(s) \int_{t_0}^s \lambda_4(\tau) w(u(\tau)) d\tau ds.$$

Then

$$u(t) \leq W^{-1} \left[W(c) + \int_{t_0}^t \left(\lambda_1(s) + \lambda_2(s) + \lambda_3(s) \int_{t_0}^s \lambda_4(\tau) d\tau \right) ds \right],$$

where $t_0 \leq t < b_1$, W , W^{-1} are the same functions as in Lemma 2.4, and

$$b_1 = \sup \left\{ t \geq t_0 : W(c) + \int_{t_0}^t \left(\lambda_1(s) + \lambda_2(s) \int_{t_0}^s \lambda_3(\tau) d\tau \right) ds \in \text{dom} W^{-1} \right\}.$$

LEMMA 2.6. [5] *Let $k, u, \lambda_1, \lambda_2, \lambda_3, \lambda_4 \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$, $u \leq w(u)$ and $w(u)$ be nondecreasing in u . Suppose that for some $c \geq 0$,*

$$u(t) \leq c + \int_{t_0}^t \lambda_1(s) \left(\int_{t_0}^s (\lambda_2(\tau) u(\tau) + \lambda_3(\tau) \int_{t_0}^{\tau} k(r) w(u(r)) dr) d\tau + \lambda_4(s) u(s) \right) ds,$$

for $t \geq t_0 \geq 0$ and for some $c \geq 0$. Then

$$u(t) \leq W^{-1} \left[W(c) + \int_{t_0}^t \lambda_1(s) \left(\int_{t_0}^s (\lambda_2(\tau) + \lambda_3(\tau) \int_{t_0}^{\tau} k(r) dr) d\tau + \lambda_4(s) \right) ds \right],$$

where $t_0 \leq t < b_1$, W , W^{-1} are the same functions as in Lemma 2.4, and

$$b_1 = \sup \left\{ t \geq t_0 : W(c) + \int_{t_0}^t \lambda_1(s) \left(\int_{t_0}^s (\lambda_2(\tau) + \lambda_3(\tau) \int_{t_0}^\tau k(r) dr) d\tau + \lambda_4(s) \right) ds \in \text{dom} W^{-1} \right\}.$$

LEMMA 2.7. [11] Let $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8 \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$, and $w(u)$ be nondecreasing in u , $u \leq w(u)$. Suppose that for some $c > 0$ and $0 \leq t_0 \leq t$,

$$u(t) \leq c + \int_{t_0}^t \lambda_1(s) u(s) ds + \int_{t_0}^t \lambda_2(s) w(u(s)) ds + \int_{t_0}^t \lambda_3(s) \int_{t_0}^s (\lambda_4(\tau) u(\tau) + \lambda_5(\tau) \int_{t_0}^\tau \lambda_6(r) w(u(r)) dr) d\tau ds + \int_{t_0}^t \lambda_7(s) \int_{t_0}^s \lambda_8(\tau) w(u(\tau)) d\tau ds.$$

Then

$$u(t) \leq W^{-1} \left[W(c) + \int_{t_0}^t \left(\lambda_1(s) + \lambda_2(s) + \lambda_3(s) \int_{t_0}^s (\lambda_4(\tau) + \lambda_5(\tau) \int_{t_0}^\tau \lambda_6(r) dr) d\tau + \lambda_7(s) \int_{t_0}^s \lambda_8(\tau) d\tau \right) ds \right],$$

where $t_0 \leq t < b_1$, W , W^{-1} are the same functions as in Lemma 2.4, and

$$b_1 = \sup \left\{ t \geq t_0 : W(c) + \int_{t_0}^t \left(\lambda_1(s) + \lambda_2(s) + \lambda_3(s) \int_{t_0}^s (\lambda_4(\tau) + \lambda_5(\tau) \int_{t_0}^\tau \lambda_6(r) dr) d\tau + \lambda_7(s) \int_{t_0}^s \lambda_8(\tau) d\tau \right) ds \in \text{dom} W^{-1} \right\}.$$

For the proof we need the following corollary from Lemma 2.7.

COROLLARY 2.8. Let $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$, and $w(u)$ be nondecreasing in u , $u \leq w(u)$. Suppose that for some $c > 0$ and $0 \leq t_0 \leq t$,

$$u(t) \leq c + \int_{t_0}^t \lambda_1(s) \int_{t_0}^s \left(\lambda_2(\tau) u(\tau) + \lambda_3(\tau) w(u(\tau)) + \lambda_4(\tau) \int_{t_0}^\tau \lambda_5(r) w(u(r)) dr \right) d\tau ds.$$

Then

$$u(t) \leq W^{-1} \left[W(c) + \int_{t_0}^t \lambda_1(s) \int_{t_0}^s \left(\lambda_2(\tau) + \lambda_3(\tau) + \lambda_4(\tau) \int_{t_0}^\tau \lambda_5(r) dr \right) d\tau ds \right],$$

where $t_0 \leq t < b_1$, W , W^{-1} are the same functions as in Lemma 2.4, and

$$b_1 = \sup \left\{ t \geq t_0 : W(c) + \int_{t_0}^t \lambda_1(s) \int_{t_0}^s (\lambda_2(\tau) + \lambda_3(\tau) + \lambda_4(\tau) \int_{t_0}^{\tau} \lambda_5(r) dr) d\tau ds \in \text{dom} W^{-1} \right\}.$$

3. Main results

In this section, we investigate the asymptotic property for solutions of nonlinear perturbed functional differential systems.

To obtain the asymptotic property, the following assumptions are needed:

(H1) The solution $x = 0$ of (2.1) is EASV.

(H2) $w(u)$ is nondecreasing in u , $u \leq w(u)$.

THEOREM 3.1. *Assume that (H1), (H2), and the perturbing term g in (2.2) satisfies*

$$(3.1) \quad |g(t, y(t), Ty(t))| \leq e^{-\alpha t} (a(t)|y(t)| + |Ty(t)|),$$

and

$$(3.2) \quad |Ty(t)| \leq b(t) \int_{t_0}^t k(s)w(|y(s)|)ds, |h(t, y(t))| \leq c(t)|y(t)|$$

where $\alpha > 0$, $a, b, c, k, w \in C(\mathbb{R}^+)$, $a, b, c, k, w \in L^1(\mathbb{R}^+)$. If

$$(3.3) \quad M(t_0) = W^{-1} \left[W(c) + \int_{t_0}^{\infty} M e^{\alpha s} \left(\int_{t_0}^s (a(\tau) + b(\tau) \int_{t_0}^{\tau} k(r) dr) d\tau + c(s) \right) ds \right] < \infty,$$

where $t \geq t_0$ and $c = |y_0| M e^{\alpha t_0}$, then all solutions of (2.2) approach zero as $t \rightarrow \infty$.

Proof. Let $x(t) = x(t, t_0, y_0)$ be the solution of (2.1) with $x(t_0, t_0, y_0) = y_0$, existing on $[t_0, \infty)$. By Lemma 2.3, any solution $y(t) = y(t, t_0, y_0)$ of (2.2) passing through (t_0, y_0) is given by

$$(3.4) \quad y(t, t_0, y_0) = x(t, t_0, y_0) + \int_{t_0}^t \Phi(t, s, y(s)) \left(\int_{t_0}^s g(\tau, y(\tau), Ty(\tau)) d\tau + h(s, y(s)) \right) ds.$$

From the assumption (H1), the solution $x = 0$ of (2.1) is EASV, and so it is EAS by Remark 2.2. Applying (3.1), (3.2), and (3.4), we have

$$\begin{aligned} |y(t)| &\leq |x(t)| + \int_{t_0}^t |\Phi(t, s, y(s))| \left(\int_{t_0}^s |g(\tau, y(\tau), Ty(\tau))| d\tau + h(s, y(s)) \right) ds \\ &\leq M|y_0|e^{-\alpha(t-t_0)} + \int_{t_0}^t Me^{-\alpha(t-s)} \left(c(s)|y(s)| + \int_{t_0}^s e^{-\alpha\tau} (a(\tau)|y(\tau)| \right. \\ &\quad \left. + b(\tau) \int_{t_0}^{\tau} k(r)w(|y(r)|) dr) d\tau \right) ds. \end{aligned}$$

It follows from (H2) that

$$\begin{aligned} |y(t)| &\leq M|y_0|e^{-\alpha(t-t_0)} + \int_{t_0}^t Me^{-\alpha(t-s)} \left(c(s)|y(s)|e^{\alpha s} + \int_{t_0}^s (a(\tau)|y(\tau)|e^{\alpha\tau} \right. \\ &\quad \left. + b(\tau) \int_{t_0}^{\tau} k(r)w(|y(r)|e^{\alpha r}) dr) d\tau \right) ds. \end{aligned}$$

Set $u(t) = |y(t)|e^{\alpha t}$. By Lemma 2.6 and (3.3) we obtain

$$\begin{aligned} |y(t)| &\leq e^{-\alpha t} W^{-1} \left[W(c) + \int_{t_0}^t Me^{\alpha s} \left(c(s) + \int_{t_0}^s (a(\tau) + b(\tau) \int_{t_0}^{\tau} k(r) dr) d\tau \right) ds \right] \\ &\leq e^{-\alpha t} M(t_0), \end{aligned}$$

where $t \geq t_0$ and $c = M|y_0|e^{\alpha t_0}$. The above estimation yields the desired result. \square

REMARK 3.2. Letting $b(t) = c(t) = 0$ in Theorem 3.1, we obtain the similar result as that of Corollary 3.8 in [5].

THEOREM 3.3. Assume that (H1), (H2), and the perturbing term g in (2.2) satisfies

$$(3.5) \quad |g(t, y(t), Ty(t))| \leq e^{-\alpha t} \left(a(t)w(|y(t)|) + |Ty(t)| \right),$$

and

$$(3.6) \quad |Ty(t)| \leq b(t) \int_{t_0}^t k(s)w(|y(s)|) ds, \quad |h(t, y(t))| \leq \int_{t_0}^t c(s)|y(s)| ds$$

where $\alpha > 0$, $a, b, c, k, w \in C(\mathbb{R}^+)$, $a, b, c, k, w \in L^1(\mathbb{R}^+)$. If

$$(3.7) \quad M(t_0) = W^{-1} \left[W(c) + \int_{t_0}^{\infty} Me^{\alpha s} \int_{t_0}^s \left(a(\tau) + c(\tau) + b(\tau) \int_{t_0}^{\tau} k(r) dr \right) d\tau ds \right] < \infty,$$

where $t \geq t_0$ and $c = |y_0|Me^{\alpha t_0}$, then all solutions of (2.2) go to zero as $t \rightarrow \infty$.

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (2.1) and (2.2), respectively. By the assumption (H1), the solution $x = 0$ of (2.1) is EASV. Therefore, it is EAS by Remark 2.2. Using (3.4), (3.5), and (3.6), we have

$$\begin{aligned} |y(t)| &\leq M|y_0|e^{-\alpha(t-t_0)} + \int_{t_0}^t Me^{-\alpha(t-s)} \int_{t_0}^s e^{-\alpha\tau} \left(a(\tau)w(|y(\tau)|) \right. \\ &\quad \left. + b(\tau) \int_{t_0}^{\tau} k(r)w(|y(r)|)dr + c(\tau)|y(\tau)| \right) d\tau ds. \end{aligned}$$

It follows from (H2) that

$$\begin{aligned} |y(t)| &\leq M|y_0|e^{-\alpha(t-t_0)} + \int_{t_0}^t Me^{-\alpha(t-s)} \int_{t_0}^s \left(c(\tau)|y(\tau)|e^{\alpha\tau} \right. \\ &\quad \left. + a(\tau)w(|y(\tau)|e^{\alpha\tau}) + b(\tau) \int_{t_0}^{\tau} k(r)w(|y(r)|e^{\alpha r})dr \right) d\tau ds. \end{aligned}$$

Set $u(t) = |y(t)|e^{\alpha t}$. By Corollary 2.8 and (3.7) we obtain

$$\begin{aligned} |y(t)| &\leq e^{-\alpha t} W^{-1} \left[W(c) + \int_{t_0}^t Me^{\alpha s} \int_{t_0}^s \left(a(\tau) + c(\tau) + b(\tau) \int_{t_0}^{\tau} k(r)dr \right) d\tau ds \right] \\ &\leq e^{-\alpha t} M(t_0), \end{aligned}$$

where $t \geq t_0$ and $c = M|y_0|e^{\alpha t_0}$. This completes the proof. \square

REMARK 3.4. Letting $b(t) = c(t) = 0$ in Theorem 3.3, we obtain the similar result as that of Theorem 3.5 in [5].

THEOREM 3.5. Suppose that (H1), (H2), and the perturbing term g in (2.2) satisfies

$$(3.8) \quad \int_{t_0}^t |g(s, y(s), Ty(s))| ds \leq e^{-\alpha t} \left(a(t)|y(t)| + |Ty(t)| \right),$$

and

$$(3.9) \quad |Ty(t)| \leq b(t) \int_{t_0}^t k(s)w(|y(s)|) ds, \quad |h(t, y(t))| \leq e^{-\alpha t} c(t)w(|y(t)|)$$

where $\alpha > 0$, $a, b, c, k, w \in C(\mathbb{R}^+)$, $a, b, c, k, w \in L^1(\mathbb{R}^+)$. If

$$(3.10) \quad M(t_0) = W^{-1} \left[W(c) + M \int_{t_0}^{\infty} \left(a(s) + c(s) + b(s) \int_{t_0}^s k(\tau) d\tau \right) ds \right] < \infty, \quad b_1 = \infty,$$

where $c = M|y_0|e^{\alpha t_0}$, then all solutions of (2.2) approach zero as $t \rightarrow \infty$.

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (2.1) and (2.2), respectively. From the assumption (H1), the solution $x = 0$ of (2.1) is EASV, and so it is EAS. By (3.4), (3.8), and (3.9), we have

$$\begin{aligned} |y(t)| &\leq M|y_0|e^{-\alpha(t-t_0)} + \int_{t_0}^t M e^{-\alpha(t-s)} \left(e^{-\alpha s} a(s)|y(s)| + e^{-\alpha s} c(s)w(|y(s)|) \right. \\ &\quad \left. + e^{-\alpha s} b(s) \int_{t_0}^s k(\tau)w(|y(\tau)|)d\tau \right) ds. \end{aligned}$$

Using the assumption (H2), we obtain

$$\begin{aligned} |y(t)| &\leq M|y_0|e^{-\alpha(t-t_0)} + \int_{t_0}^t M e^{-\alpha t} \left(a(s)|y(s)|e^{\alpha s} + c(s)w(|y(s)|e^{\alpha s}) \right) ds \\ &\quad + \int_{t_0}^t M e^{-\alpha t} b(s) \int_{t_0}^s k(\tau)w(|y(\tau)|e^{\alpha \tau})d\tau ds. \end{aligned}$$

Set $u(t) = |y(t)|e^{\alpha t}$. Then, it follows from Corollary 2.5 and (3.10) that

$$\begin{aligned} |y(t)| &\leq e^{-\alpha t} W^{-1} \left[W(c) + M \int_{t_0}^t \left(a(s) + c(s) + b(s) \int_{t_0}^s k(\tau)d\tau \right) ds \right] \\ &\leq e^{-\alpha t} M(t_0), \end{aligned}$$

where $t \geq t_0$ and $c = M|y_0|e^{\alpha t_0}$. From the above estimation, we obtain the desired result. \square

REMARK 3.6. Letting $b(t) = c(t) = 0$ in Theorem 3.5, we obtain the similar result as that of Corollary 3.8 in [5].

THEOREM 3.7. Suppose that (H1), (H2), and the perturbing term g in (2.2) satisfies

$$(3.11) \quad \int_{t_0}^t |g(s, y(s), Ty(s))| ds \leq e^{-\alpha t} \left(a(t)w(|y(t)|) + |Ty(t)| \right),$$

and

$$(3.12) \quad |Ty(t)| \leq b(t) \int_{t_0}^t k(s)w(|y(s)|) ds, |h(t, y(t))| \leq e^{-\alpha t} c(t)|y(t)|,$$

where $\alpha > 0$, $a, b, c, k, w \in C(\mathbb{R}^+)$, $a, b, c, k, w \in L^1(\mathbb{R}^+)$. If

$$(3.13) \quad M(t_0) = W^{-1} \left[W(c) + M \int_{t_0}^{\infty} \left(a(s) + c(s) + b(s) \int_{t_0}^s k(\tau)d\tau \right) ds \right] < \infty,$$

where $b_1 = \infty$ and $c = M|y_0|e^{\alpha t_0}$, then all solutions of (2.2) go to zero as $t \rightarrow \infty$.

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (2.1) and (2.2), respectively. By the assumption (H1), the solution $x = 0$ of (2.1) is EASV. Hence, it is EAS. Applying (3.4), (3.11), and (3.12), we have

$$\begin{aligned} |y(t)| &\leq M|y_0|e^{-\alpha(t-t_0)} + \int_{t_0}^t M e^{-\alpha(t-s)} \left(e^{-\alpha s} c(s)|y(s)| + e^{-\alpha s} a(s)w(|y(s)|) \right. \\ &\quad \left. + e^{-\alpha s} b(s) \int_{t_0}^s k(\tau)w(|y(\tau)|)d\tau \right) ds. \end{aligned}$$

From the assumption (H2), we obtain

$$\begin{aligned} |y(t)| &\leq M|y_0|e^{-\alpha(t-t_0)} + \int_{t_0}^t M e^{-\alpha t} \left(c(s)|y(s)|e^{\alpha s} + a(s)w(|y(s)|e^{\alpha s}) \right) ds \\ &\quad + \int_{t_0}^t M e^{-\alpha t} b(s) \int_{t_0}^s k(\tau)w(|y(\tau)|e^{\alpha \tau})d\tau ds. \end{aligned}$$

Set $u(t) = |y(t)|e^{\alpha t}$. Then, it follows from Corollary 2.5 and (3.13) that

$$\begin{aligned} |y(t)| &\leq e^{-\alpha t} W^{-1} \left[W(c) + M \int_{t_0}^t \left(a(s) + c(s) + b(s) \int_{t_0}^s k(\tau)d\tau \right) ds \right] \\ &\leq e^{-\alpha t} M(t_0), \end{aligned}$$

where $t \geq t_0$ and $c = M|y_0|e^{\alpha t_0}$, and so the proof is complete. \square

REMARK 3.8. Letting $c(t) = 0$ in Theorem 3.7, we obtain the same result as that of Theorem 3.7 in [12].

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References

- [1] V. M. Alekseev, An estimate for the perturbations of the solutions of ordinary differential equations, Vestn. Mosk. Univ. Ser. I. Math. Mekh. **2** (1961), 28-36(Russian).
- [2] F. Brauer, *Perturbations of nonlinear systems of differential equations*, J. Math. Anal. Appl. **14** (1966), 198-206.
- [3] S. I. Choi and Y. H. Goo, *h-stability and boundedness in perturbed functional differential systems*, Far East J. Math. Sci(FJMS) **97** (2015), 69-93.

- [4] S. I. Choi and Y. H. Goo, *Lipschitz and asymptotic stability for nonlinear perturbed differential systems*, J. Chungcheong Math. Soc. **27** (2014), 591-602.
- [5] S. I. Choi and Y. H. Goo, *Lipschitz and asymptotic stability of nonlinear systems of perturbed differential equations*, Korean J. Math. **23** (2015), 181-197.
- [6] S. K. Choi, Y. H. Goo, and N. J. Koo, *Lipschitz and exponential asymptotic stability for nonlinear functional systems*, Dynamic Systems and Applications **6** (1997), 397-410.
- [7] S. K. Choi, N. J. Koo, and S. M. Song, *Lipschitz stability for nonlinear functional differential systems*, Far East J. Math. Sci(FJMS)I **5** (1999), 689-708.
- [8] S. Elaydi and H. R. Farran, *Exponentially asymptotically stable dynamical systems*, Appl. Anal. **25** (1987), 243-252.
- [9] P. Gonzalez and M. Pinto, *Stability properties of the solutions of the nonlinear functional differential systems*, J. Math. Appl. **181** (1994), 562-573.
- [10] Y. H. Goo, *Lipschitz and asymptotic stability for perturbed nonlinear differential systems*, J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. **21** (2014), 11-21.
- [11] Y. H. Goo, *Uniform Lipschitz stability and asymptotic behavior for perturbed differential systems*, Far East J. Math. Sci(FJMS) **99** (2016), 393-412.
- [12] Y. H. Goo and Y. Cui, *Uniform Lipschitz and asymptotic stability for perturbed differential systems*, J. Chungcheong Math. Soc. **26** (2013), 831-842.
- [13] V. Lakshmikantham and S. Leela, *Differential and Integral Inequalities: Theory and Applications Vol. I*, Academic Press, New York and London, 1969.
- [14] B. G. Pachpatte, *Stability and asymptotic behavior of perturbed nonlinear systems*, J. Math. Anal. Appl. **16** (1974), 14-25.

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